

Quasinormal modes of the polytropic hydrodynamic vortex

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Analogue systems are a powerful instrument to investigate and understand in a controlled setting many general-relativistic effects. Here, we focus on superradiant-triggered instabilities and quasinormal modes. We consider a compressible hydrodynamic vortex characterized by a polytropic equation of state, the *polytropic hydrodynamic vortex*, a purely circulating system with an ergoregion but no event horizon. We compute the quasinormal modes of this system numerically with different methods, finding excellent agreement between them. When the fluid velocity is larger than the speed of sound, an ergoregion appears in the effective spacetime, triggering an “ergoregion instability.” We study the details of the instability for the polytropic vortex, and in particular find analytic expressions for the marginally-stable configuration.

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I. INTRODUCTION

Black holes are an important component of our Universe, thought to play a role in the dynamics of galaxies and in star formation. They have also come to play a prominent role in high-energy physics, and even fundamental physics as they are simultaneously an “elementary particle” of gravitation and an entity where classical and quantum effects are intertwined through Hawking radiation. Unfortunately, many of these features and properties have the undesirable consequence that black holes are “hard to see”. It is therefore very convenient to use analogue setups where these properties are present, but which can be manipulated in the laboratory. One such example are acoustic holes, idealized models in fluid dynamics mimicking the behavior of a curved spacetime [1–3].

Acoustic holes have, for instance, been used to understand wave scattering in more general curved spacetimes [4–8] as well as the impact of UV cutoffs in the Hawking radiation in reasonable completions of General Relativity [9]. One specially interesting feature of curved spacetimes in General Relativity is the ergoregion, bounded by an infinite-redshift surface which represents the static limit where it’s impossible to remain at rest with respect to distant inertial observers. Negative-energy states are possible within the ergoregion, leading to superradiant effects if horizons are present and to instabilities otherwise [10].

The purpose of this work is to explore the phenomenology of superradiant instabilities in analogue models. We

focus on the hydrodynamic vortex [11–13], a purely circulating compressible system, and we will work with a polytropic equation of state [14].

Here we show that the polytropic hydrodynamic vortex is unstable under linearized perturbations, and that the appearance of these instabilities is directly related to the presence of an ergoregion and absence of an event horizon, the so-called ergoregion instability [10, 15]. This result is a generalization of a previous result (obtained for an incompressible system in [16]) for compressible systems that satisfy a polytropic equation of state and that describe a wide-class of thermodynamical processes, as, e.g., isentropic processes [17–20]. Here we focus on values of polytropic index N_p describing isentropic processes (also adiabatic processes), with a compatible experimental setup in perfect gases.

The remainder of this paper is structured as follows. In Sec. II we describe the spacetime of the polytropic hydrodynamic vortex. In Sec. III we study the perturbations of the polytropic hydrodynamic vortex using descriptions in the time and frequency domains. In Sec. IV we obtain the quasinormal mode (QNM) frequencies of the polytropic hydrodynamic vortex using the method of lines (MOL), direct integration (DI) and the continued fraction (CF) method. In Sec. V we validate and comment our results comparing the QNM frequencies obtained via MOL, DI and CF methods. Furthermore, we investigate the static (marginally-stable) resonances of the polytropic hydrodynamic vortex, studying this system in the regime between stability and instability. We conclude with a brief discussion in Sec. VI.

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II. THE POLYTROPIC HYDRODYNAMIC VORTEX

The (effective) spacetime of the polytropic hydrodynamic vortex is produced by an irrotational, barotropic and purely circulating fluid characterized by a polytropic equation of state. The line element of this system may be written as

$$ds^2 = \frac{\rho}{c_s} \left[-c_s^2 dt^2 + (rd\theta - v_\theta dt)^2 + dr^2 + dz^2 \right], \quad (1)$$

where ρ is the mass density, v_θ is the angular component of the flow velocity \vec{v} , i.e., $\vec{v} = v_\theta \hat{\theta}$, and c_s is the speed of sound, which may be defined as

$$c_s \equiv \sqrt{\frac{dP}{d\rho}}, \quad (2)$$

assuming that the fluid is barotropic, i.e.,

$$P = P(\rho), \quad (3)$$

where P is the hydrostatic pressure. Note that the quantities ρ , P , \vec{v} and c_s are given by the local properties of the unperturbed fluid flow [2].

We may obtain expressions for ρ , c_s and v_θ assuming that the fluid flow is irrotational

$$\nabla \times \vec{v} = 0, \quad (4)$$

and that it satisfies the Euler equation, i.e.,

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{2} \nabla v^2 + \frac{\nabla P}{\rho} = 0. \quad (5)$$

Considering that density ρ , pressure P and angular flow velocity v_θ are functions of the radial coordinate only, i.e., $\rho = \rho(r)$, $P = P(r)$, $v_\theta = v_\theta(r)$, we obtain, respectively, from Eqs. (4) and (5), the following expressions

$$r \frac{dv_\theta}{dr} + v_\theta = 0, \quad (6a)$$

$$\frac{v_\theta^2}{r} - \frac{1}{\rho} \frac{dP}{dr} = 0. \quad (6b)$$

From Eqs. (6), it follows that

$$v_\theta = \frac{C}{r}, \quad (7)$$

$$\frac{dP}{dr} - \frac{C^2}{r^3} \rho = 0, \quad (8)$$

where C is a constant related to the circulation of the fluid [13].

We may solve Eq. (8) using an expression that denotes the relation between P and ρ , i.e., an equation of state [cf. Eq. (3)]. Here we use a polytropic equation of state, namely

$$P = k_p \rho^{1+1/N_p}, \quad (9)$$

where k_p is the polytropic constant and N_p is a constant called polytropic index [19].

Certain thermodynamical processes, in which one of quantities of the system remains constant, can be described by specific values of the polytropic index N_p [19, 20], namely: (i) isobaric (constant pressure) processes, described by polytropic index $N_p = -1$; (ii) isometric (constant volume) and isopycnic (constant density) processes, described by polytropic index $N_p = 0$; (iii) isothermal (constant temperature) processes, described by polytropic index $N_p = \pm\infty$; and (iv) isentropic (constant entropy) processes, described by polytropic index $N_p = 1/(\lambda - 1)$. The quantity λ is the so-called *specific heat ratio*, being given by $\lambda = c_p/c_v$, where c_p and c_v are, respectively, the specific heat at constant pressure and the specific heat at constant volume, for a perfect gas [19].

Substituting Eq. (9) into Eq. (8), we obtain the following first-order differential equation:

$$k_p \left(1 + \frac{1}{N_p} \right) \rho^{1/N_p} \frac{d\rho}{dr} - \frac{C^2}{r^3} \rho = 0, \quad (10)$$

whose solution may be written as

$$\rho(r) = \rho_\infty \left(1 - \frac{r_c^2}{r^2} \right)^{N_p}, \quad (11)$$

where ρ_∞ is the density at $r \rightarrow \infty$. Note that the density goes to zero and becomes non-physical at a critical radius [21] defined by

$$r_c \equiv \frac{|C|}{\sqrt{2K_p(N_p + 1)}}. \quad (12)$$

Here we defined $K_p \equiv k_p \rho_\infty^{1/N_p}$. Therefore, the polytropic hydrodynamic vortex has an essential singularity at the critical radius $r = r_c$, denoting that this spacetime, as well as Kerr spacetime, has a singularity with non-pointlike structure [18].

Using the equation of state given by Eq. (9) and the definition of the speed of sound given by Eq. (2), we may obtain an expression to the speed of sound, as a function of the density, for a polytropic fluid, i.e.,

$$c_s = \sqrt{k_p \left(1 + \frac{1}{N_p} \right) \rho^{1/N_p}}. \quad (13)$$

Using Eqs. (11) and (13), we obtain get

$$c_s(r) = c_{s\infty} \sqrt{1 - \frac{r_c^2}{r^2}}, \quad (14)$$

where $c_{s\infty}$ is the speed of sound at $r \rightarrow \infty$, given by

$$c_{s\infty} = \sqrt{K_p \left(1 + \frac{1}{N_p} \right)}. \quad (15)$$

As it happens to the density [cf. Eq. (11)], the speed of sound [cf. Eq. (14)] becomes zero and non-physical at the critical radius r_c [21] [see Eq. (12)].

The local Mach number is defined as being the ratio between absolute value of the flow velocity and speed of sound, i.e., $M \equiv |\vec{v}|/c_s$. Using the local Mach number as a parameter, we may define a supersonic (subsonic) flow as $M > 1$ ($M < 1$). For $M = 1$ (or, explicitly, $|\vec{v}| = c_s$), we may write the outer boundary of the ergoregion, r_e , as

$$r_e \equiv \frac{|C|\sqrt{2N_p + 1}}{\sqrt{2K_p(N_p + 1)}}. \quad (16)$$

As seen from line element given by Eq. (1), the polytropic hydrodynamic vortex has no coordinate singularities, i.e., it has no event horizon, but it has an ergoregion with outer boundary at $r = r_e$.

Dividing Eq. (16) by Eq. (12) we find the following ratio between the radius of the outer boundary of the ergoregion r_e and critical radius r_c :

$$\frac{r_e}{r_c} = \sqrt{2N_p + 1}. \quad (17)$$

From Eq. (17), for $N_p \geq 0$, we find that $r_e \geq r_c$, i.e., the outer boundary of the ergoregion r_e is located outside the outer boundary of the critical radius r_c . For $-1/2 \leq N_p < 0$, the outer boundary of the ergoregion r_e is located inside of the outer boundary of the critical radius r_c , i.e., $r_e < r_c$. For $N_p < -1/2$, the outer boundary of the ergoregion r_e is complex, and therefore non-physical.

Basically, we are interested in values of the polytropic index such that r_e is real and $r_e > r_c$, i.e., $N_p > 0$. As seen in Eq. (12), the case in which $N_p = 0$ gives us $r_c \rightarrow 0$, therefore we will not consider this case here. The $N_p = 0$ case [which describe an isopycnic (constant density) processes] was analyzed in Ref. [16]. Here we focus our attention in the case of the polytropic index describing isentropic processes.

Essentially, an isentropic process is always an adiabatic process too, i.e., in addition to the entropy of the system remain constant, the process occurs without transfer of heat energy throughout [19]. The main motivation to study isentropic (adiabatic) processes here is based on fact that the propagation of sound waves in a gas is fundamentally an adiabatic process [22, 23].

We now assume the polytropic hydrodynamic vortex to be composed by a perfect gas, described by the equation [19]

$$P = \frac{RT}{\mu}\rho, \quad (18)$$

where $R = 8.314462 \text{ J K}^{-1} \text{ mol}^{-1}$ is the perfect gas constant, T is the temperature of the gas, and μ is the mean molecular weight of the gas. In this case, from Eqs. (9), (11) and (18), we may write expressions for the

pressure and temperature of the gas, as functions of r , given, respectively, by

$$P(r) = P_\infty \left(1 - \frac{r_c^2}{r^2}\right)^{N_p+1}, \quad (19)$$

where P_∞ is the pressure at $r \rightarrow \infty$, and

$$T(r) = T_\infty \left(1 - \frac{r_c^2}{r^2}\right), \quad (20)$$

where T_∞ is the temperature at $r \rightarrow \infty$. Pressure and temperature, as the density and speed of sound, becomes zero and non-physical at critical radius r_c [21] (see Eq. (12)).

In Table I we exhibit estimates of the polytropic index N_p , of the constant K_p [which, using Eqs. (18) and (20), may be written as $K_p = RT_\infty/\mu$], of the critical radius r_c , of the density at infinity ρ_∞ , of the speed of sound at infinity $c_{s\infty}$, and of the ratio $c_{s\infty}/r_c$, for some gases.

In Table II we exhibit estimates of the density $\rho(r)$, speed of sound $c_s(r)$, pressure $P(r)$ and temperature $T(r)$ for the Ethane gas (C_2H_6) at different positions of the radial coordinate.

In Fig. 1 we plot, respectively, fluid density ρ , speed of sound c_s , pressure P and temperature T , as functions of r and for different values of polytropic index N_p .

III. PERTURBATIONS OF A POLYTROPIC HYDRODYNAMIC VORTEX

Linear perturbations propagating in acoustic space-times are described by the Klein-Gordon equation, namely

$$\square\Phi = \frac{1}{\sqrt{|g|}}\partial_a\left(\sqrt{|g|}g^{ab}\partial_b\Phi\right) = 0, \quad (21)$$

where g^{ab} is the contravariant *effective metric*, given by

$$g^{ab} = -\frac{1}{\rho c_s} \begin{pmatrix} 1 & 0 & C/r^2 & 0 \\ 0 & -c_s^2 & 0 & 0 \\ C/r^2 & 0 & (C^2/r^2 - c_s^2)/r^2 & 0 \\ 0 & 0 & 0 & -c_s^2 \end{pmatrix}$$

and $g \equiv \det(g_{ab}) = -\rho^4 r^2 / c_s^2$.

The spacetime of the polytropic hydrodynamic vortex is cylindrically symmetric, so that we may decompose the field Φ in terms of azimuthal harmonics, namely

$$\Phi(t, r, \theta, z) = \frac{1}{\sqrt{r}} \sum_{m=-\infty}^{\infty} \psi_m(t, r) \exp(im\theta), \quad (22)$$

where m is an integer number that is related with the angular momentum of the perturbation. The coordinate

TABLE I. Estimates of the polytropic index N_p , of the constant K_p , of the critical radius r_c , of the density at infinity ρ_∞ , speed of sound at infinity $c_{s\infty}$, and of the ratio $c_{s\infty}/r_c$, for some gases, obtained considering the temperature $T_\infty = 288.15$ K and the pressure $P_\infty = 101325$ Pa (estimates of N_p , K_p and μ were obtained from data extracted from Ref. [24]). Here we choose the circulation $C = 0.5$ m²/s. In parentheses are indicated the units and the order of magnitude of each quantity.

Gas	N_p	K_p (10^4 m ² /s ²)	r_c (10^{-4} m)	ρ_∞ (kg/m ³)	$c_{s\infty}$ (m/s)	$c_{s\infty}/r_c$ (10^5 s ⁻¹)
Argon (Ar)	1.49	5.99853	9.14814	1.68916	316.613	3.46095
Nitrogen (N ₂)	2.47	8.55036	6.49080	1.18504	346.584	5.33962
Carbon dioxide (CO ₂)	3.28	5.44379	7.32457	1.86129	266.524	3.63876
Ethane (C ₂ H ₆)	4.54	7.96745	5.32158	1.27174	311.808	5.85931

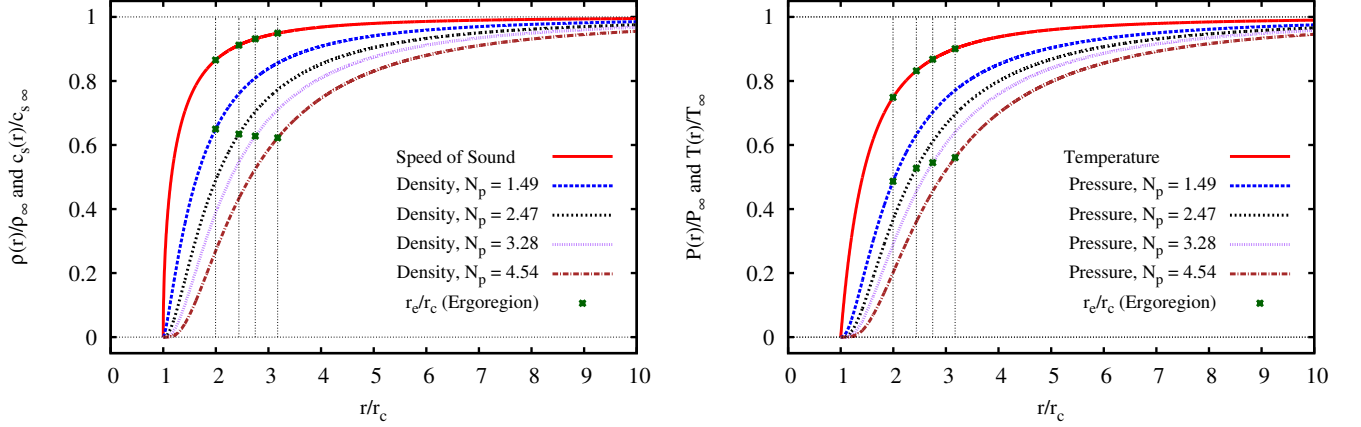


FIG. 1. (Left plots) Fluid density and speed of sound as functions of r , for polytropic index $N_p = 1.49$, $N_p = 2.47$, $N_p = 3.28$ and $N_p = 4.54$. (Right plots) Pressure and temperature as functions of r , for the same choices of the polytropic index.

TABLE II. Estimates of the density $\rho(r)$, speed of sound $c_s(r)$, pressure $P(r)$ and temperature $T(r)$ for the Ethane gas (C₂H₆) at $r/r_c = 2.0$, $r/r_c = 4.0$, $r/r_c \rightarrow \infty$. In parentheses are indicated the units used in each quantity.

Quantity	$r/r_c = 2.0$	$r/r_c = 4.0$	$r/r_c \rightarrow \infty$
ρ (kg/m ³)	0.34448	0.94873	1.27174
c_s (m/s)	270.033	301.906	311.808
P (Pa)	20585.2	70865.9	101325
T (K)	216.112	270.141	288.150

z is trivial, and the spacetime under study is effectively three-dimensional.

Substituting the covariant metric g^{ab} and Eq. (22) into Eq. (21), we obtain

$$\left[-\frac{1}{c_s^2} \left(\frac{\partial}{\partial t} + \frac{iCm}{r^2} \right)^2 + \frac{\partial^2}{\partial r^2} + \frac{1}{\rho} \frac{d\rho}{dr} \frac{\partial}{\partial r} - \frac{1}{r^2} \left(m^2 - \frac{1}{4} \right) - \frac{1}{2r\rho} \frac{d\rho}{dr} \right] \psi_m(t, r) = 0. \quad (23)$$

In the frequency domain, we assume single-frequency

modes $\psi_m(t, r)$ by using the following ansatz

$$\psi_m(t, r) = u_{\omega m}(r) \exp(-i\omega t). \quad (24)$$

Substituting Eq. (24) into Eq. (23), we obtain

$$\left[\frac{d^2}{dr^2} + \frac{1}{\rho} \frac{d\rho}{dr} \frac{d}{dr} + \frac{1}{c_s^2} \left(\omega - \frac{Cm}{r^2} \right)^2 - \frac{1}{r^2} \left(m^2 - \frac{1}{4} \right) - \frac{1}{2r\rho} \frac{d\rho}{dr} \right] u_{\omega m}(r) = 0. \quad (25)$$

Furthermore, we may use Eqs. (11), (12), (14), and (15) into Eq. (23), to obtain the following partial differential equation

$$\left[-\left(x^2 \frac{\partial}{\partial \tau} + \frac{i\sqrt{2N_p}Cm}{|C|} \right)^2 + x^2(x^2 - 1) \frac{\partial^2}{\partial x^2} + 2xN_p \frac{\partial}{\partial x} - (x^2 - 1) \left(m^2 - \frac{1}{4} \right) - N_p \right] \psi_m(\tau, x) = 0, \quad (26)$$

where we defined a dimensionless time coordinate τ , given by

$$\tau \equiv tc_{s\infty}/r_c, \quad (27)$$

and a dimensionless radial coordinate x , namely

$$x \equiv r/r_c. \quad (28)$$

Equivalently, using Eqs. (11), (12), (14), and (15) into Eq. (25), we obtain the following ordinary differential equation for single-frequency modes $u_{\varpi m}(x)$

$$\left[x^2 (x^2 - 1) \frac{d^2}{dx^2} + 2xN_p \frac{d}{dx} + \left(x^2 \varpi - \frac{\sqrt{2N_p} C m}{|C|} \right)^2 - (x^2 - 1) \left(m^2 - \frac{1}{4} \right) - N_p \right] u_{\varpi m}(x) = 0, \quad (29)$$

where we defined a dimensionless frequency ϖ , given by

$$\varpi \equiv r_c \omega / c_{s\infty}. \quad (30)$$

The ordinary differential equation given by Eq. (29) has regular singular points at the origin $x = 0$, at the critical radius $x = x_c \equiv 1$, and at $x = -1$, and an irregular singular point at infinity ($x \rightarrow \infty$).

From Eq. (29), it is straightforward to see that the polytropic hydrodynamic vortex has the following symmetries in the frequency ϖ , related to azimuthal number m and circulation C

$$\varpi(m, C) = -\varpi^*(-m, C) = -\varpi^*(m, -C) = \varpi(-m, -C),$$

where “*” denotes complex conjugation. Henceforth, taking into account these symmetries, we assume without loss of generality that $m > 0$ and $C > 0$.

A. Boundary and initial conditions

We consider two different boundary conditions at $x_{\min} \equiv r_{\min}/r_c$, adopting essentially the same physically acceptable boundary conditions proposed in Ref. [16] (i.e., we consider a cylinder with radius $r = r_{\min}$ made of a certain material with acoustic impedance Z), namely: (i) BCI, a boundary condition of Dirichlet type which mimics low- Z materials, and (ii) BCII, a boundary condition of Neumann type which mimics high- Z materials [25].

The boundary condition BCI may be defined as

$$[u_{\varpi m}(x)]_{x=x_{\min}} = 0. \quad (31)$$

The boundary condition BCII is given by (cf. Eq. (22))

$$\left[\frac{d}{dx} \left(\frac{u_{\varpi m}(x)}{\sqrt{x}} \right) \right]_{x=x_{\min}} = 0. \quad (32)$$

Both these boundary conditions are realistic and whether BCI or BCII hold in practice depends on how the experimental apparatus is implemented in the laboratory.

At large radial distances we require outgoing Sommerfeld or causal boundary conditions, which in the frequency domain, and given our choice of the Fourier transform, amount to

$$u_{\varpi m}(x \rightarrow \infty) \sim \exp(i\varpi x). \quad (33)$$

As initial condition to time-domain analysis, we use the following Gaussian package

$$[\psi_m(\tau, x)]_{\tau=0} = (x - x_{\min})^2 \exp\left(\frac{-(x - x_0)^2}{2\sigma^2}\right), \quad (34)$$

and its first order derivative with respect to the time

$$\left[\frac{\partial \psi_m}{\partial \tau}(\tau, x) \right]_{\tau=0} = 0, \quad (35)$$

where x_0 is the position of the center of the peak of the Gaussian function (middle point), and σ sets the width of the Gaussian function. Note that the Gaussian package, given by Eq. (34), satisfies both boundary conditions BCI and BCII.

Our main task consists therefore in studying solutions of Eqs. (26) and (29), subjected to these boundary and initial conditions.

IV. NUMERICAL METHODS

A. Method of lines (MOL)

We may determine the time-domain profiles associated with the QNMs of the polytropic hydrodynamic vortex applying a numerical method to solve the partial differential equation (26). We study the evolution of a Gaussian disturbance in the time domain, using the method of lines (MOL) as a numerical simulation [26, 27]. The MOL involves a second-order spatial coordinate discretization and fourth-order Runge-Kutta method to advance in time [28–30].

To apply the MOL in Eq. (26), we discretize the radial coordinate $x \rightarrow x_j = x_{\min} + jh$ (for a range $x_{\min} \leq x \leq x_{\max}$), the wave function $\psi_m(\tau, x) \rightarrow \psi_j$, first-order spatial derivative $\frac{\partial}{\partial x} \psi_m(\tau, x) \rightarrow \xi_j$, with

$$\xi_j = \frac{1}{2h} (\psi_{j+1} - \psi_{j-1}) + \mathcal{O}(h^2); \quad (36)$$

and second-order spatial derivative $\frac{\partial^2}{\partial x^2} \psi_m(\tau, x) \rightarrow \chi_j$, namely,

$$\chi_j = \frac{1}{h^2} (\psi_{j+1} - 2\psi_j + \psi_{j-1}) + \mathcal{O}(h^2), \quad (37)$$

where h is the grid spacing [26, 30].

Employing the discretizations and transforming the spatial second-order derivative using the definition $\zeta_j \equiv \frac{d\psi_j}{d\tau}$, we may obtain from Eq. (26) a set of two first-order differential equations. Then, we apply the fourth-order Runge-Kutta method in each first-order differential equation to evolve the Gaussian perturbation (34) in time domain (for a range $0 \leq \tau \leq \tau_{\max}$) [28].

B. Direct integration (DI) method

In the frequency-domain, the differential equation (29) subjected to the boundary conditions described in Section III A is an eigenvalue problem for the frequency ω . Frequencies that satisfy the eigenvalue problem are called QNM frequencies. To investigate superradiant instabilities, we compute the QNM frequencies for the polytropic hydrodynamic vortex. QNMs are generically modes with complex frequencies. The time-dependence of the fluctuations, expressed in Eq. (24), implies that if the imaginary part of the QNM frequencies is negative ($\text{Im}[\omega] < 0$) the spacetime is stable and the perturbations vanish for late times as $\psi \sim \exp(-|\text{Im}[\omega]|t)$. In contrast, if the imaginary part of the QNM frequencies is positive ($\text{Im}[\omega] > 0$), the perturbations are amplified as $\psi \sim \exp(\text{Im}[\omega]t)$ and the spacetime is unstable.

We may solve directly the differential equation (29) using a direct integration (DI) method to obtain the QNM frequencies. The DI is based on the shooting method and numerical root-finding to obtain frequencies in the complex domain [31]. We write the outgoing solution at infinity as a generalized power series

$$u_{\varpi m}(x \rightarrow \infty) \sim \exp(i\varpi x) \sum_{i=0} \frac{b_i}{x^i}. \quad (38)$$

The series (38) and its first-order derivative are then used as boundary conditions to directly integrate Eq. (29) inwards for a range $x_{\max} \geq x \geq x_{\min}$.

The QNM frequencies are the roots of this integration procedure that satisfy the appropriate BCI or BCII at $x = x_{\min}$, i.e., Eqs. (31) and (32), respectively. To find these roots, we use standard root-finding algorithms such as Newton's method.

C. Continued fraction (CF) method

An alternative to directly integrating Eq. (29), consists in expressing the problem as a continued fraction (CF) to find the QNM frequencies [32]. We may define the Frobenius-like series in the neighborhood of $x = x_{\min}$, namely

$$u_{\varpi m}(x) = \exp(i\varpi x) \sum_{n=0} a_n \left(1 - \frac{x_{\min}}{x}\right)^n. \quad (39)$$

Substituting Eq. (39) into Eq. (29), we find the following five-term recurrence relation:

$$\alpha_0 a_2 + \beta_0 a_1 + \gamma_0 a_0 = 0, \quad (40a)$$

$$\alpha_1 a_3 + \beta_1 a_2 + \gamma_1 a_1 + \delta_1 a_0 = 0, \quad (40b)$$

$$\alpha_n a_{n+2} + \beta_n a_{n+1} + \gamma_n a_n + \delta_n a_{n-1} + \epsilon_n a_{n-2} = 0, \quad (40c)$$

for $n \geq 2$,

where the recurrence coefficients are given by

$$\alpha_n = 4(1+n)(2+n)(-1+x_{\min}^2), \quad (41a)$$

$$\beta_n = 8(1+n)[N_p - n(-2+x_{\min}^2) + i(-1+x_{\min}^2)(i+x_{\min}\varpi)], \quad (41b)$$

$$\gamma_n = 4[m^2 + 2m^2 N_p - 6nN_p + 2i(2n+N_p)x_{\min}\varpi + x_{\min}^2(-m^2 + n + n^2 - 2\sqrt{2}m\sqrt{N_p}\varpi + \varpi^2)] - 1 - 24n^2 - 4N_p + x_{\min}^2, \quad (41c)$$

$$\delta_n = 2[5 - 12n + 8n^2 - 8N_p + 12nN_p - 4m^2(1+2N_p) - 4i(-1+n+N_p)x_{\min}\varpi], \quad (41d)$$

$$\epsilon_n = -(-3+2n)(-3+2n+4N_p) + m^2(4+8N_p). \quad (41e)$$

Using a double Gaussian elimination (cf. Refs. [16, 33]) from the five-term recurrence relation (40) we may write the following three-term recurrence relation:

$$\alpha_n a_{n+2} + \beta_n a_{n+1} + \gamma_n a_n = 0, \quad \text{for } n \geq 0. \quad (42)$$

The recurrence coefficients α_n, β_n and γ_n are complex functions that depend on the frequency ϖ , the azimuthal number m , the polytropic index N_p and x_{\min} .

For BCI, considering $a_0 = 0$ in Eq. (42), we obtain the following continued-fraction

$$\beta_0 - \frac{\alpha_0 \gamma_1}{\beta_1 - \frac{\alpha_1 \gamma_2}{\beta_2 - \frac{\alpha_2 \gamma_3}{\beta_3 - \dots}}} = 0. \quad (43)$$

From Eq. (39) and considering $n \rightarrow n-1$ in Eq. (42) we find the following relation for BCII

$$1 - 2i\varpi x_{\min} + \frac{2\gamma_1}{\beta_1 - \frac{\alpha_1 \gamma_2}{\beta_2 - \frac{\alpha_2 \gamma_3}{\beta_3 - \dots}}} = 0. \quad (44)$$

Eqs. (43) and (44) can be solved with standard root-finding algorithms such as Newton's method.

V. RESULTS

Concerning the radial position r_{\min} where we impose boundary conditions BCI and BCII, the QNMs of the polytropic hydrodynamic vortex may be separated in two categories:

- (i) QNMs for r_{\min} located outside of the outer boundary of the ergoregion r_e , which in dimensionless radial coordinate can be written as $x_{\min} \geq \sqrt{2N_p + 1}$ [see Eq. (17)].
- (ii) QNMs for r_{\min} located inside of the outer boundary of the ergoregion r_e , $1 < x_{\min} < \sqrt{2N_p + 1}$.

For r_{\min} located outside the outer boundary r_e of the ergoregion, the polytropic hydrodynamic vortex admits only stable QNM frequencies; whereas for r_{\min} located inside r_e , the polytropic hydrodynamic vortex admits both stable and unstable QNM frequencies [16]. Here we are especially interested in computing the QNM frequencies

for r_{\min} located inside r_e , which exhibit the ergoregion instabilities associated with this system.

We follow standard conventions of ordering the QNM frequencies ϖ (in dimensionless units) by their imaginary part [34]. The fundamental mode is the one with largest imaginary component $\text{Im}[\varpi]$. Thus, if the mode is unstable ($\text{Im}[\varpi] > 0$), the fundamental mode corresponds to the smallest instability timescale, and for stable ($\text{Im}[\varpi] < 0$) modes it corresponds to the longest-lived mode.

A. Boundary conditions imposed outside of the outer boundary of the ergoregion

We have computed QNM frequencies for r_{\min} located outside of the outer boundary r_e of the ergoregion, checking the agreement between three methods for different values of the polytropic index N_p . Examples are shown in Table III.

In Table III we exhibit the estimates of the QNM frequencies ϖ obtained via MOL, DI and CF methods for different values polytropic index N_p . We have obtained excellent agreement between three methods.

TABLE III. QNM frequencies ϖ for azimuthal number $m = 3$, and $x_{\min} = 4.0$, obtained numerically from estimates via MOL, DI and CF methods, for polytropic index $N_p = 1.49$, $N_p = 2.47$, $N_p = 3.28$ and $N_p = 4.54$ using boundary conditions BCI and BCII. For MOL, we compute QNM frequencies from time-domain profiles $|Re(\psi_m(\tau, x))|$ extracted at $x = 12.0$, via Eq. (26). Here we use a grid spacing $h = 1/1000$, width of Gaussian function $\sigma = 0.25$, and middle point of Gaussian function $x_0 = 6.0$. Note that the QNM frequency ϖ is dimensionless, for conversion into dimensional frequency ω , one should multiply by $c_{s\infty}/r_c$ (cf. the data in Table I).

N_p	Method	BCI		BCII	
		$\text{Re}(\varpi)$	$\text{Im}(\varpi)$	$\text{Re}(\varpi)$	$\text{Im}(\varpi)$
1.49	MOL	-0.277610	-0.230845	-0.373637	-0.110406
	DI	-0.277939	-0.235285	-0.374032	-0.109901
	CF	-0.277323	-0.232678	-0.374032	-0.109901
2.47	MOL	-0.262089	-0.191755	-0.322674	-0.080349
	DI	-0.262375	-0.192064	-0.322619	-0.080100
	CF	-0.262819	-0.191891	-0.322619	-0.080100
3.28	MOL	-0.251992	-0.166754	-0.290045	-0.061799
	DI	-0.252594	-0.166545	-0.289911	-0.061835
	CF	-0.252738	-0.166635	-0.289911	-0.061835
4.54	MOL	-0.238283	-0.136121	-0.248886	-0.040595
	DI	-0.238832	-0.135947	-0.248850	-0.040702
	CF	-0.238817	-0.135995	-0.248850	-0.040702

B. Boundary conditions imposed inside of the outer boundary of the ergoregion

Next, we choose r_{\min} located inside of the outer boundary r_e of the ergoregion and outside of the critical radius r_c ($1 < x_{\min} < \sqrt{2N_p + 1}$).

In Table IV we exhibit the estimates of the QNM frequencies ϖ for azimuthal number $m = 5$, obtained via MOL, DI and CF methods, for different values of polytropic index N_p .

TABLE IV. QNM frequencies ϖ for azimuthal number $m = 5$, obtained numerically from estimates via MOL, DI and CF methods, for polytropic index $N_p = 2.47$, $N_p = 3.28$ and $N_p = 4.54$. We impose boundary conditions BCI and BCII at $x_{\min} = 2.0$. For MOL, we compute QNM frequencies from time-domain profiles $|Re(\psi_m(\tau, x))|$ extracted at $x = 10.0$, via Eq. (26). We use a grid spacing $h = 1/1000$, width of Gaussian function $\sigma = 0.25$, and middle point of Gaussian function $x_0 = 4.0$.

N_p	Method	BCI	
		$\text{Re}(\varpi)$	$\text{Im}(\varpi)$
2.47	MOL	-0.536536	-0.024658
	DI	-0.536540	-0.024618
	CF	-0.536540	-0.024618i
3.28	MOL	-0.380340	-0.001655
	DI	-0.380328	-0.001675
	CF	-0.380328	-0.001675
4.54	DI	-0.123292	-5.329260×10^{-9}
	CF	-0.123292	-5.328963×10^{-9}

N_p	Method	BCII	
		$\text{Re}(\varpi)$	$\text{Im}(\varpi)$
2.47	DI	-0.136950	-1.172176×10^{-9}
	CF	-0.136950	-1.172059×10^{-9}
3.28	DI	+0.143251	$+3.268528 \times 10^{-11}$
	CF	+0.143251	$+3.268469 \times 10^{-11}$
4.54	DI	+0.519010	$+7.000436 \times 10^{-7}$
	CF	+0.519010	$+7.000436 \times 10^{-7}$

C. Ergoregion instability of the polytropic hydrodynamic vortex

In Fig. 2 we plot real and imaginary parts of the fundamental ($n = 0$) QNM frequencies ϖ , for different values of azimuthal numbers m and polytropic index $N_p = 4.54$, obtained via CF method, imposing boundary conditions BCI and BCII at different values of x_{\min} . Analyzing the imaginary part of the QNM frequencies ϖ , we conclude that, as the azimuthal number m increases, the threshold between stability and instability (i.e., in the neighborhood of $\text{Im}(\varpi) = 0$) increases, tending to the outer

boundary r_e of the ergoregion, which in dimensionless radial coordinate may be represented by $x_{\min} = x_e$ (where $x_e = r_e/r_c$). This behavior can be seen more clearly in the zooms for BCI and BCII exhibited in the bottom plots of Fig. 2, being most prominent to boundary conditions BCII than BCI.

As an example of a possible experimental implementation, we estimate the QNM frequencies ω for a vortex made of Ethane gas (whose polytropic constant $K_p = 7.96745 \times 10^4 \text{ m}^2/\text{s}^2$ and polytropic index $N_p = 4.54$). We may obtain the QNM frequencies ω using Eq. (30), and the data exhibited in Tables I, II and IV. To this particular experimental setup, we consider the circulation $C = 0.5 \text{ m}^2/\text{s}$, subject to a perturbation with azimuthal number $m = 5$ and we impose boundary conditions of Neumann type (BCII) at $r_{\min} = 2.0r_c$. The real and imaginary parts of the fundamental ($n = 0$) QNM frequency ω are then, respectively, $\text{Re}(\omega) = 0.304104 \text{ MHz}$ and $\text{Im}(\omega) = 0.410177 \text{ s}^{-1}$. From the imaginary part it is possible to estimate the instability timescale of this perturbation, which is $t_{\text{scale}} \equiv 1/\text{Im}(\omega) = 2.43797 \text{ s}$.

D. Static (marginally-stable) resonances

Now, we investigate a critical configuration of marginal-stability of the polytropic hydrodynamic vortex, referred to as static (marginally-stable) resonances [35, 36], mathematically represented by $\text{Re}(\omega) = \text{Im}(\omega) = 0$. (It may be seen in Fig. 2 that for specific values of x_{\min} the QNMs frequencies goes to zero.) An analytical study of a particular case of the incompressible hydrodynamic vortex ($N_p = 0$), in its marginally-stable regime, was addressed in Ref. [35]. Here we study a compressible system characterized by the polytropic equation of state (8).

Considering $\omega = 0$, we may obtain the following analytical solution for the ordinary differential equation (29):

$$u_{0m}(x) = a_1 a^+ f^-(x) + a_2 a^- f^+(x), \quad (45)$$

where a_1 and a_2 are integration constants, $a^\pm = i^{7/2 \mp m} x^{1/2 \pm m}$,

$$f^\pm(x) = {}_2F_1\left(\frac{N_p - b \pm m}{2}, \frac{N_p + b \pm m}{2}; 1 \pm m; x^{-2}\right), \quad (46)$$

with ${}_2F_1[\alpha, \beta, \gamma, y]$ being the hypergeometric function [37] and $b = \sqrt{N_p^2 + m^2(1 + 2N_p)}$.

We assume that $a_1 = 0$, so that the physically acceptable solution (45) is finite at $x \rightarrow \infty$. It follows that

$$u_{0m}(x) = a_2 a^- f^+(x). \quad (47)$$

Applying the boundary conditions, BCI [given by Eq. (31)] and BCII [given by Eq. (32)] in the solution

given by Eq. (47), we may write the following equations

$$[f^+(x)]_{x=x_{\min}^{(n)}} = 0 \quad (\text{for BCI}), \quad (48)$$

$$\left[\frac{mf^+}{x} - \frac{df^+}{dx}\right]_{x=x_{\min}^{(n)}} = 0 \quad (\text{for BCII}), \quad (49)$$

respectively, where $x_{\min}^{(n)}$ is the n th positive root of Eqs. (48) (for BCI) and (49) (for BCII) and describe the static (marginally-stable) resonances of the polytropic hydrodynamic vortex. To find these roots, we use a standard root-finding algorithms such as Newton's method.

In Table V we exhibit the estimates of $x_{\min}^{(0)}$ (where $n = 0$ denotes the fundamental QNM frequencies) for polytropic index $N_p = 4.54$ and for different values azimuthal number m , obtained numerically. Note that as the azimuthal number m increases, the values of $x_{\min}^{(0)}$ tend to the values of the outer boundary x_e of the ergoregion (being most prominent to $m = 1000$ and to boundary conditions BCII than BCI). The results exhibited in Table V are in excellent agreement with the data extracted from Fig. 2, this may be denoted when we compare with the results for $x_{\min}^{(0)}$ in parentheses, which are obtained via CF method (some are exhibited in Fig. 2), with the results obtained from Eqs. (48) and (49).

TABLE V. Estimates of $x_{\min}^{(0)}$ [the 0th root of Eqs. (48) (for BCI) and (49) (for BCII)], obtained numerically using a root-finding algorithm, for polytropic index $N_p = 4.54$, and for azimuthal number $m = 6, 7, 8, 9, 10, 100, 1000$, using boundary conditions BCI and BCII. The results exhibited in parentheses were obtained via CF method (some are exhibited in Fig. 2).

m	$x_{\min}^{(0)}$ (BCI)	$x_{\min}^{(0)}$ (BCII)
6	1.991469762231 (1.991)	2.506392063437 (2.506)
7	2.069839310556 (2.070)	2.565087771362 (2.565)
8	2.135952625074 (2.136)	2.612175436164 (2.612)
9	2.192645873342 (2.193)	2.650971380111 (2.651)
10	2.241922364228 (2.242)	2.683607543289 (2.684)
100	2.926472047450 (2.926)	3.061894259945 (3.061)
1000	3.118751713938 (3.118)	3.150204534073 (3.150)
$x_e(N_p = 4.54) = 3.174901573278$		

In Fig. 3 we plot the estimates of $\bar{x} \equiv r_{\min}^{(0)}/r_e$, as a function of polytropic index N_p (for the azimuthal number $m = 10$) and as a function of azimuthal number m (for the polytropic index $N_p = 4.54$). Essentially, the growth of the polytropic index N_p implies in a decrease of \bar{x} . On the other hand, the growth of the azimuthal number m implies in a growth of \bar{x} . For large values of the polytropic index N_p , \bar{x} becomes constant ($\bar{x} < 1$) and for large values of the azimuthal number m , $\bar{x} \rightarrow 1$, denoting that the static resonances ($\omega = 0$) occur at $r_{\min} = r_e$, for large values of the azimuthal number m .

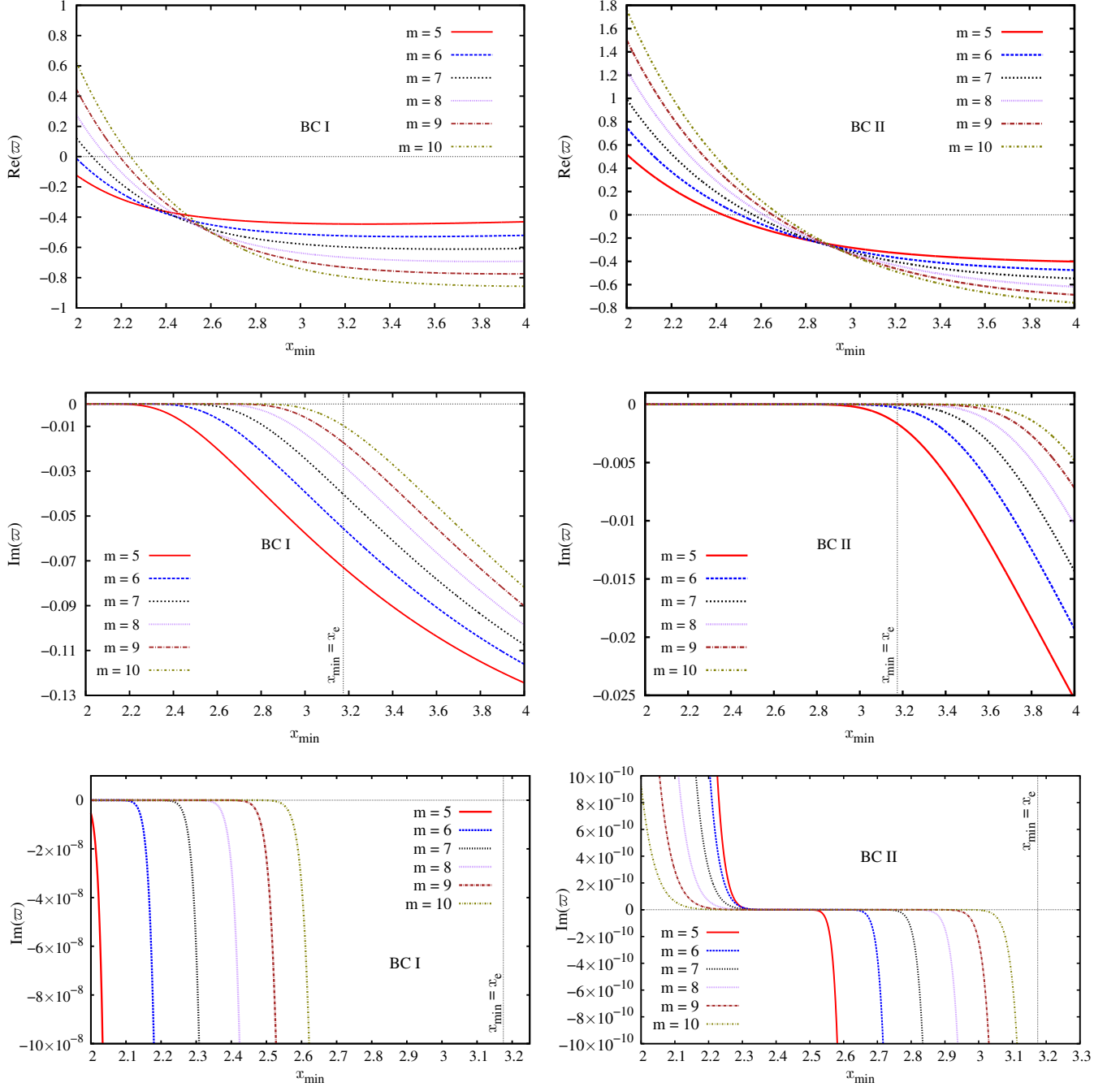


FIG. 2. Real (top plots) and imaginary (middle and bottom plots) parts of the fundamental ($n = 0$) QNM frequencies ϖ , as a function of x_{\min} , for polytropic index $N_p = 4.54$ and different values of azimuthal numbers m , obtained via CF method. Here we imposed boundary conditions BC I (left) and BC II (right) at $x = x_{\min}$. The dotted vertical lines at $x_{\min} = x_e$, in the middle and bottom plots, correspond to the position of the outer boundary r_e of the ergoregion. Note that the QNM frequency ϖ and radial position x_{\min} are dimensionless. For conversion to the frequency ω , one should multiply ϖ by $5.85931 \times 10^5 \text{ s}^{-1}$ and to obtain the radial position r_{\min} , one should multiply x_{\min} by $5.32158 \times 10^{-4} \text{ m}$ (cf. the data in Table I).

VI. CONCLUSION

Here we used the method of lines (MOL), direct integration (DI) and continued-fraction (CF) methods to obtain the QNM frequencies of the polytropic hydrody-

namic vortex, an effective spacetime produced by purely circulating compressible fluid that satisfies a polytropic equation of state. To validate our results, we compared the QNM frequencies obtained via these three different methods and obtained excellent agreement. Further-

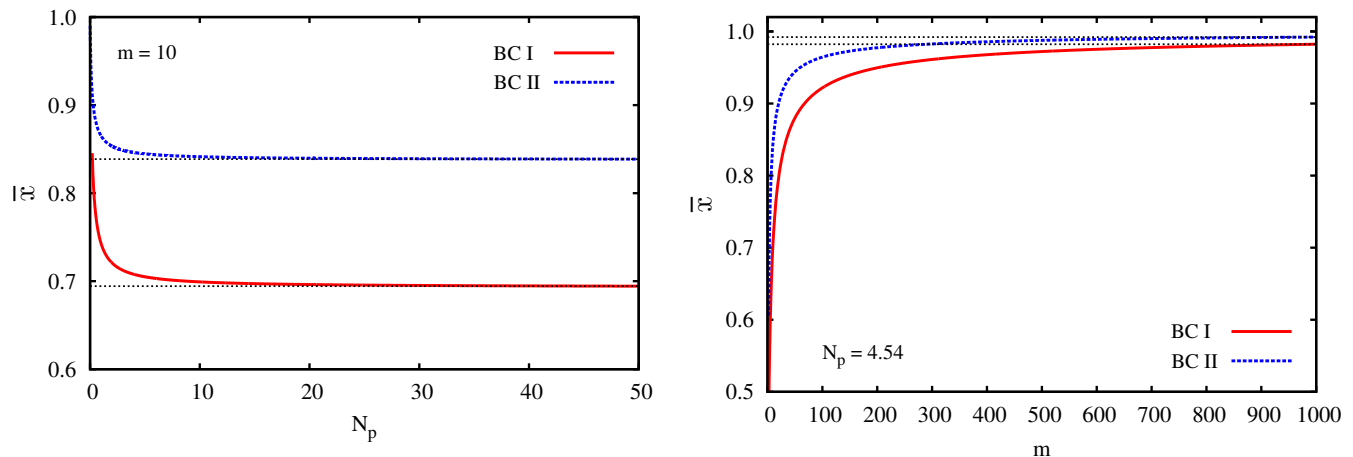


FIG. 3. *Left plots:* Estimates of \bar{x} , as a function of polytropic index N_p and for azimuthal number $m = 10$, for the boundary conditions BC I and BC II. *Right plots:* Estimates of \bar{x} , as a function of azimuthal number m and polytropic index $N_p = 4.54$, for the boundary conditions BC I and BC II. The results were obtained numerically using a root-finding algorithm.

more, we studied the polytropic hydrodynamic vortex in the regime between stability and instability, obtaining the configuration of the parameters of this system (azimuthal number m and polytropic index N_p) required for the onset of the ergoregion instability. We focused our attention on the dependence of the QNM frequencies with the polytropic index N_p . As the polytropic index N_p increases, the magnitude of real and imaginary parts of the QNM frequencies increase (decrease) for unstable (stable) modes. Furthermore, we have shown that the polytropic hydrodynamic vortex, a compressible system with an ergoregion and without an event horizon is unstable. Together with the instability of the corresponding incompressible system, namely, the hydrodynamic vortex studied in Ref. [16], this establishes the ergoregion instability as a generic phenomena. The instability is more clearly revealed by computing QNMs, imposing boundary conditions at $r = r_{\min}$ located inside the outer boundary r_e of the ergoregion. Finally, we have shown that the onset of ergoregion instability approaches the outer boundary of the ergoregion as the azimuthal number m increases, denoting that for large values of the azimuthal number m the QNMs become unstable, independently of the configuration (circulation C and polytropic index N_p) of the system (as may be seen in Fig. 2). Furthermore, we have shown that for large values of azimuthal number

m , the static resonances ($\omega = 0$) occur at $r_{\min} = r_e$ (as may be seen in Table V and Fig. 3).

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